Nature of a topological quantum phase transition in a chiral spin liquid model

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We study the finite temperature nature of a quantum phase transition between an Abelian and a non-Abelian topological phase in an exactly solvable model of a chiral spin liquid [1]. By virtue of the exact solvability, this model can serve as a testbed for developing better measures for describing topological quantum phase transitions. We characterize this phase transition in terms of the global flux and entanglement entropy, and discuss to what extent the existence of a topologically ordered ground state with non-Abelian excitations is revealed at finite temperature.

Characterizing and detecting topological order is one of the central questions in the field of topological phases. The challenge lies in that these new type of quantum ground states are not associated with any local broken symmetry. Of broader interest in the context of quantum phase transitions(QPT) is a question of the nature of a quantum critical point when a system enters a topologically ordered phase [2]. In a conventional QPT, in which a local order parameter starts to gain an expectation value at the quantum critical point (QCP), the nature of the QCP is of ultimate significance. Even though it is effectively a point of measure zero, it governs a much larger phase space often called the "quantum critical region" (See Fig.1(a) where we denote the expectation value of an order parameter by $\langle\phi\rangle$). One can ask if, and to what extent, an analogy holds for topological quantum phase transitions. For such a question, we need a formulation and understanding of measures of topological order at finite temperature.

Since Wen and Niu [3] coined the term "topological order" in association with the ground state degeneracy of (Abelian) fractional quantum Hall (FQH) states on topologically non-trivial surfaces, such ground state degeneracy has been widely used as an indicator of topological order including the non-Abelian FQH states [4]. Further, the implications of such degeneracy on fractionalization has also been discussed[5, 6]. However, the extension of this indicator, which is defined at $T = 0$ and not directly accessible experimentally, to a measure at finite temperature is an open question.

More recently, the concept of "topological entanglement entropy" has been gaining interest as an indicator of topological order [7, 8] or a topological QCP [9]. The corresponding quantity at finite temperature has also been studied [10]. However, one of the issues with topological entanglement entropy is that it does not always distinguish phases with obviously different topological orders such as weak pairing (vortices follow non-Abelian statistics) and strong pairing (vortices are Abelian) $p + ip$ superconductors [11]. This shows that the information about the ground state is significantly condensed upon mapping to a single entropic quantity.

We start with the observation that a chiral spin liquid (CSL) model [1] can provide an ideal testbed for developing a better understanding of a topological QPT by playing the role that the transverse field Ising chain played in the study of conventional QPTs. It has the virtue of being exactly solvable, and exhibiting a non-trivial QPT between non-Abelian(nA) and Abelian(A) phases analogous to the weak pairing and strong pairing limits of a $p + ip$ superconductor; the same physics can also be accomplished for the honeycomb model with three-spin interaction [14]. We take a twofold approach: First, we employ the notion of an "expectation value" of a global flux operator introduced by Nussinov and Ortiz [12] as a finite temperature extension of the concept of ground state degeneracy. Second, we contrast this result to what can be learned from entanglement entropy.

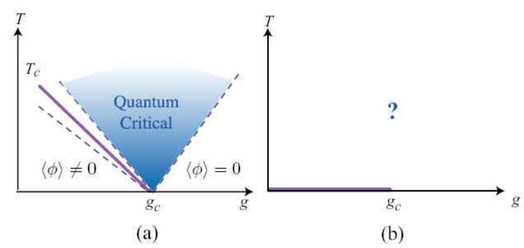

FIG. 1: Phase diagrams in $g - T$ phase space. (a)A typical QPT phase diagram for conventional order with the quantum fluctuations controlled by tuning parameter g. Here we sketched a case with the dynamical critical exponent $z = 1$. (b)Topological QPT phase diagram.

Model– The exactly solvable CSL model on the star lattice [1] is a variant of a spin model with topological order first introduced by Kitaev[13] on the honeycomb lattice. In this variation, ground states spontaneously break time reversal symmetry and a QPT between A and nA phases is accessible through the exact solution. For brevity we employ a Majorana fermion representation of the model. We represent spin-1/2 Pauli operators σ_i^α ($\alpha = x, y, z$) of the original spin model at each lattice site i[1] by four species of Majorana fermions c_i and d_i^α, $\sigma_i^\alpha = ic_i d_i^\alpha$ under the constraint

$$D_i \equiv c_i d_i^x d_i^y d_i^z = 1 \tag{1}$$

so that $\sigma^x\sigma^y\sigma^z = i$ as is expected of spin 1/2 operators. In terms of these Majorana fermions, the Hamiltonian is

$$\mathcal{H}[\{\hat{U}_{ij}\}] = J \sum_{x,y,z-\mathrm{link}} \hat{U}_{ij} i c_i c_j + J' \sum_{x',y',z'-\mathrm{link}} \hat{U}_{ij} i c_i c_j, \quad (2)$$

where $\hat{U}_{ij} \equiv -i d_i^\alpha d_j^\alpha$ is defined at each α and α' bonds between sites (i, j) (see Fig.2) and acts as a $\mathbb{Z}_2$ gauge field living on the ij bond [18]. As $\hat{U}_{ij}$ has no dynamics ($[\hat{U}_{ij}, \mathcal{H}] = 0$) it can be replaced by a set of $\mathbb{Z}_2$ variables $u_{ij} = \pm 1$ reducing Eq. (2) to a quadratic Hamiltonian $\mathcal{H}[\{u_{ij}\}]$ parameterized by $\{u_{ij}\}$. For a loop L, the Z_2 flux is given by $\phi_L = \prod_{ij\in L} u_{ij} = \pm 1$.

Defining $g \equiv J'/J$, $\mathcal{H}[\{u_{ij}\}]$ can be diagonalized as

$$\mathcal{H}[\{u_{ij}\}] = J \sum_{n,\vec{k}} \epsilon_{n,\vec{k}}[\{u_{ij}\}; g]\, (b^\dagger_{n,\vec{k}} b_{n,\vec{k}} - 1/2), \quad (3)$$

by finding the complex fermion operators $b_{n,\vec{k}}$ that are linear in c_i's for momentum $\vec{k}$ and band index $n = 1, 2, 3$ (there are six sites per unit cell in the Majorana fermion Hamiltonian). This yields the entire spectrum $\epsilon_{n,\vec{k}}[\{u_{ij}\}; g]$. The ground states are uniform flux states with $\phi^0_L = -1$ for all 12-plaquettes and $\phi^0_L = 1$ or -1 for all triangular plaquettes, spontaneously breaking time reversal symmetry. A vortex on a plaquette L, defined by $\phi_L = -\phi^0_L$, costs finite energy for all g. Moreover, the uniform flux ground state is degenerate on a torus and *the topological degeneracy changes* across the nA to A QPT at $g_c = \sqrt{3}$[1]. However, care is needed for discerning physical states that satisfy the constraint Eq. (1) for each configuration of $\{u_{ij}\}$.

Topological degeneracy and the projection operator– A clue towards an extension of topological degeneracy to finite temperature lies in the g-dependent effects of the constraint Eq. (1). The constraint defines the *physical* states of the free fermion Hamiltonian Eq. (3) and is sensitive to g. Eq. (1) can be implemented using a projection operator $\hat{P} = \prod_i \frac{1}{2}(1 + D_i)$ since $D_i\hat{P} = \hat{P}$ [1, 13, 16]. Moreover, $\hat{P}$ commutes with the original Hamiltonian $\mathcal{H}[\{\hat{U}_{ij}\}]$. We can show that whether a state survives projection only depends on the fermion parity, defined as $P_f = \prod_{ij\in x',y'z'\text{-links}} i c_i c_j$, and the parity of the number of vortex excitations [19] on triangle plaquettes.

In the uniform flux sector, all physical states have even fermion parity $P_f = 1$, which is particularly important for determining the topological degeneracy of the ground states on a torus. Topological degeneracy comes from the identical free fermion spectra, in the thermodynamic limit, in the four possible topological sectors, distinguished by the choice of the $\mathbb{Z}_2$ global flux:

$$\Phi_\alpha \equiv \prod_{\langle ij\rangle \in \Gamma_\alpha} u_{ij} = \pm 1, \quad (4)$$

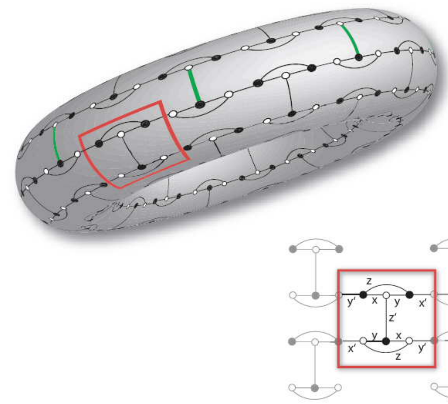

FIG. 2: A decorated brick wall lattice that is topologically equivalent to the star lattice of Ref.[1], on the surface of a torus. Green links denote u_{ij} configurations contributing to a global flux threading. The inset defines α links for 'triangles' and α' links connecting 'triangles' with $\alpha = x, y, z$. This labelling of links specifies components of spins interacting across the links in the original spin model.

where $\alpha = x, y$ label two global cycles Γ_x and Γ_y. Now $(\Phi_x, \Phi_y) = (\pm 1, \pm 1)$ are four distinct states where $\Phi_x = -1$ and $\Phi_y = -1$ indicates π flux threaded through the distinct holes of the torus (see Fig.2). Normally the fermion parity in the unprojected ground state wave function in all four topological sectors is even and independent of g. Thus, they will survive the projection and give rise to four-fold topological degeneracy. This is indeed the case on the A side ($g > g_c$). However, the fermion parity in the unprojected ground state wave function in the $(-1, -1)$ sector on the nA side ($g < g_c$) is odd and consequently it does not survive projection [20]. Thus there is only a three-fold topological ground state degeneracy in the nA phase. In summary, for A phases with uniform flux, all the physical states consist of an even number of fermionic quasiparticle excitations above the ground states in all four topological sectors. For nA phases with uniform flux, all physical states have an even number of fermion excitations in sectors $(1, 1)$, $(1, -1)$, and $(-1, 1)$, but an odd number of fermion excitations in the sector $(-1, -1)$. This has consequences not only for the topological ground state degeneracy, but also at finite temperature, as shown below.

Global flux expectation value– Motivated by the connection between the change in the allowed physical spectrum at the topological QPT and the global flux states, we consider the 'expectation value' of the global flux $\langle\Phi_\alpha\rangle$ defined as [12]

$$\langle\Phi_\alpha(T)\rangle \equiv \frac{1}{\mathcal{Z}} \mathrm{tr}\ \Phi_\alpha e^{-\mathcal{H}/T} \quad (5)$$

in a finite size system with N sites. This ties the topological degeneracy to the spectrum and offers a natural finite T extension of topological degeneracy. If we further

restrict ourselves to uniform flux states (which is valid at $T = 0$ and is a good approximation in the vicinity of the QCP where the fermion gap vanishes but vortex gap is finite), Eq. (5) can be recast as

$$\langle \Phi_\alpha \rangle = \frac{\sum_{\Phi_x, \Phi_y = \pm 1} \Phi_\alpha \mathcal{Z}^{(\Phi_x, \Phi_y)}}{\mathcal{Z}}, \qquad (6)$$

where we have defined a sub-partition function for each global flux sector (Φ_x, Φ_y). So, for the uniform flux states $\mathcal{Z}^{(\Phi_x,\Phi_y)} = \mathrm{tr}\big|_{(\Phi_x,\Phi_y)} \exp(-\mathcal{H}/T)$.

Clearly, in the absence of a dependence of the *physical* spectrum on the global flux, all the sub-partition functions will be identical $\mathcal{Z}^{(-1,-1)} = \mathcal{Z}^{(1,1)} = \mathcal{Z}^{(1,-1)} = \mathcal{Z}^{(-1,1)}$ and $\langle \Phi_\alpha \rangle$ will average out to be identically zero. This is the case for the A phase; and the case of toric code previously studied[12]. However, for the nA phase of the CSL model, the $(-1,-1)$ sector is projected out of the ground state Hilbert space and hence $\mathcal{Z}^{(-1,-1)}(T = 0)_{\mathrm{nA}} = 0$. This yields a *finite* and definite $\langle \Phi_\alpha \rangle$ in the nA phase at $T = 0$:

$$\langle \Phi_x \rangle (T = 0) = \begin{cases} 1/3 & (\mathrm{nA}, g < g_c) \\ 0 & (\mathrm{A}, g > g_c) \end{cases}. \qquad (7)$$

The significance of Eq.(7) is that $\langle \Phi_x \rangle$ is the first identification of a quantity that can be defined in a thermodynamic sense that *changes* at the topological QPT. Now the relation $\langle \Phi_x \rangle (T = 0)$ can be related to the topological degeneracy through

$$n_{DEG} = 4 - 3\langle \Phi_x \rangle (T = 0). \qquad (8)$$

Most importantly, this identification allows one to extend the notion of topological degeneracy to *finite temperature* through $\langle \Phi_x \rangle (T \neq 0)$ and to investigate the vicinity of the topological QPT that is largely unknown in Fig.1(b).

Analogous to $T_c(g)$ (the solid line of Fig.1(a)) and the cross over line (dashed line in Fig.1(a)) in the vicinity of a conventional QCP, we define and investigate a cross over temperature scale $T^*(g)$ above which $\langle \Phi_x \rangle$ falls off to zero for a system with finite size. As shown in Fig. (3), a crossover temperature scale $T^*(g)$ can be defined as the temperature scale at which $\langle \Phi_x \rangle (T)$ falls off exponentially from its zero temperature value at a given value of $g < g_c$. In Fig. (3), we show $\langle \Phi_x \rangle (T)$ for $g = 1.3$ as defined in Eq.(5). For all $g < g_c$, $\langle \Phi_x \rangle$ is nearly a constant (1/3) for $T < T^*(g)$, but decays exponentially to zero at higher temperatures. We have defined $T^*(g)$ as the point at which $\mathcal{Z}^{(-1,-1)}(T^*)/\mathcal{Z}^{(1,1)}(T^*) = e^{-1}$ by convention. Above T^*, the distinction between the A and the nA phase vanishes. The plot of $T^*(g)$ in Fig.4 shows that $T^*(g)$ is a distinct scale which is non-vanishing at the QCP $g = g_c$, unlike the excitation gap which vanishes. The excitation gap is non-vanishing in both phases where, in contrast, $T^*(g)$ is only non-zero for $g < g_c$

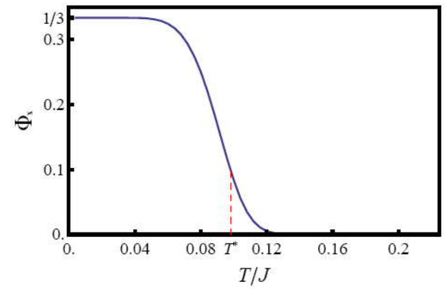

FIG. 3: Defining T^* through the exponential decay of the $\langle \Phi_x(T) \rangle$. The plot is for $g = 1.3 < g_c$ on a 60×40 lattice.

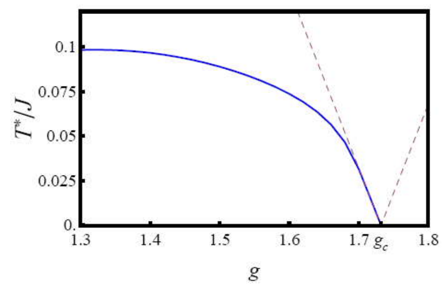

FIG. 4: The T^*(solid line) compared to the single fermion excitation gap (dashed line) for a 60×40 lattice in the vicinity of g_c. Although both Abelian and non-Abelian phases are gapped, T^* is defined only in the non-Abelian phase.

which allows the identification of the nA phase at finite temperature.

The crossover scale T^* has an intriguing system size dependence. For a large system of N sites,

$$T^* \sim \frac{\Delta(g)}{\ln N}, \qquad (9)$$

where $\Delta(g)$ is the energy gap in the fermion spectrum(see Fig.4). Hence $T^* \to 0$ as $N \to \infty$, but at a rate slower than any other quantity in the system. Therefore, the distinction between the A phase and the nA phase, while strictly vanishing at finite temperature in the thermodynamic limit, can be meaningful in some sizable range of N. Interestingly, this system size dependence bares similarity with the crossover scale for the finite temperature topological entanglement entropy of the toric code in Ref.[10]. It is also reminiscent of the finite temperature behavior of an Ising chain of finite length. An Ising chain does not order at any finite temperature in the thermodynamic limit. However, one can define a finite crossover temperature scale in a finite size system of length N by comparing the energy cost $2J$ for a domain wall and the entropic gain of $T \log N$ for N possible choice for the position of the domain wall. Nevertheless, the present size dependence Eq.(9) is rather a consequence of $\langle \Phi_x \rangle (T)$ being bounded from below by $\tanh^N(\Delta/2T)/3$ (see SOM)

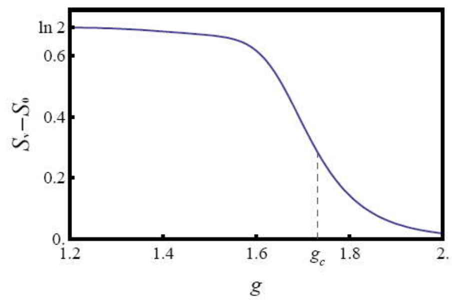

FIG. 5: The entanglement entropy change due to a vortex pair excitation for 30 × 30 Kagomé lattice sites on a torus. The change at g_c is not sharp for this finite size, though it approaches the thermodynamic limit value of $\ln 2$ in the nA phase.

and it is unrelated to local fluctuations. Possible connections underlying these apparent similarities and their implications for a "quantum critical region" is an open question.

Towards finite T entanglement entropy–While the topological entanglement entropy [7, 8] of the ground state wave function has been widely adopted as a measure of topological order, it is does not distinguish the A phase from the nA phase in the present case [17]. The topological entanglement entropy γ is defined as the universal constant term in the entanglement entropy in addition to the usual term proportional to the perimeter of the boundary: $S_{\rm ent} = \alpha L - \gamma$. Further it is known that γ is given by the total quantum dimension of the topological phase : $\gamma = \ln\sqrt{\sum_\alpha d_\alpha^2}$. Since $\{d_\alpha\} = \{1, \sqrt{2}, 1\}$ in the nA phase while $\{d_\alpha\} = \{1, 1, 1, 1\}$ in the A phase [17], the topological QPT does not affect γ which is equal to 2 in both phases. However, it is possible a finite temperature extension of this quantity might offer a possible distinction between the two phases, as excitations with distinct statistics would contribute.

An extension of the entanglement entropy to $T \neq 0$ must involve the inclusion of thermal excitations. For instance, the extension proposed in Ref.[10] (an alternative definition was proposed in Ref.[15]) retains the basic form $S_{\rm ent} = -{\rm Tr}(-\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}})$ but uses for the $\rho_{\mathcal{A}}$, a thermalized reduced density matrix

$$\rho_{\mathcal{A}}(T) = \sum_\lambda \frac{e^{-E_\lambda/T}}{Z} {\rm Tr}_{\mathcal{B}} |\Phi_\lambda\rangle\langle\Phi_\lambda|, \qquad (10)$$

where $|\Phi_\lambda\rangle$'s are energy eigenstates. Clearly Eq.(10) reduces to the usual definition at $T = 0$ when only the ground state(s) enter the sum.

While the zero temperature topological entanglement entropies of the A and nA phases are identical, the different excitations of each phase will generically lead to different finite-temperature quantities. As a first pass through the problem, we consider the change in the entanglement entropy in the presence of a pair of vortex excitations, one each in the two regions $\mathcal{A}$ and $\mathcal{B}$ whose entanglements are under consideration. Fig. (5) shows the result as a function of g. The additional entropy of $\log 2$ of the nA phase reflects the double degeneracy associated with Majorana fermion vortex core states responsible for the non-Abelian statistics of the vortices. It is clear that the characteristics of finite-energy excitations are important qualities of the topological phases.

Closing remarks– We studied the nature of the topological QPT between a nA phase and an A phase in an exactly solvable model using finite temperature extensions of two separate measures of topological order. The expectation value of the global flux $\langle\Phi\rangle(T)$ is a finite temperature extension of the ground state topological degeneracy which clearly changes at the QPT. We found that $\langle\Phi\rangle(T)$ retains the $T = 0$ value for $g < g_c$ up to a crossover temperature scale which decays logarithmically with the system size. Whether this type of crossover is ubiquitous for topological phases in two spatial dimension is an open question. As a step towards a finite temperature extension of γ, which is independent of g at $T = 0$, we considered the effect of a pair of vortices. This indicates the possibility that $\gamma(T)$ might distinguish the nA phase from the A phase.

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[18] Note that the original representation in Ref. 1 introduces $\mathbb{Z}_2$ gauge fields only on z, z', and the 'cut' links, which corresponds to a particular gauge choice here.
[19] After gauge fixing, the $\hat{P}$ expansion of Ref. [16] can be factorized: $\hat{P} = [1 + \prod_i D_i]\hat{G} = [1 + P_f \prod_{L\in\triangle,\triangledown} \phi_L]\hat{G}$, where $\hat{G} = [1+\sum_j D_j + \sum_{i<j} D_i D_j + \cdots]$ (note $D_i^2 = 1$).
[20] The exclusion of the $(-1,-1)$ state from the ground state for $g < g_c$ is tied to the nA statistics of the vortices. A global flux threading Φ_α is equivalent to the procedure of (i)creating a vortex pair (ii) transporting one vortex around the loop Γ_α(iii) annihilating the pair[6]. The $(-1,-1)$ sector is equivalent to two vortex loops linked to each other, which cannot be undone in the nA phase.

THE SUPPORTING ONLINE MATERIAL: THE SYSTEM SIZE DEPENDENCE OF T^* IN THE NON-ABELIAN PHASE.

Since the vortex gap never closes for all values of g, we can ignore the vortex excitation in calculating $\langle\Phi_x\rangle$ in the vicinity of the QPT:

$$\langle\Phi_x\rangle(T) = \frac{\mathcal{Z}_{CSL}^{(1,1)} - \mathcal{Z}_{CSL}^{(-1,-1)}}{3\mathcal{Z}_{CSL}^{(1,1)} + \mathcal{Z}_{CSL}^{(-1,-1)}}. \tag{11}$$

Since in the nA phase, the fermion occupation number parity is odd in the $(\Phi_x, \Phi_y) = (-1,-1)$ sector and even in all the other sectors

$$\mathcal{Z}_{CSL}^{(1,1)} \pm \mathcal{Z}_{CSL}^{(-1,-1)} = 2\exp(-E_G/T)\prod_{n,\vec{k}}[1 \pm \exp(-\epsilon_{n,\vec{k}}/T)], \tag{12}$$

where E_G is the ground state energy. With Eq.(12), we can recast the global flux thermal expectation value $\langle\Phi_x\rangle(T)$ as

$$\langle\Phi_x\rangle(T) = \frac{\prod_{n,\vec{k}} \tanh(\epsilon_{n,\vec{k}}/2T)}{2 + \prod_{n,\vec{k}} \tanh(\epsilon_{n,\vec{k}}/2T)}. \tag{13}$$

This implies

$$\frac{1}{3}\left[\tanh\frac{\Delta(g)}{2T}\right]^N < \langle\Phi_x\rangle < \frac{1}{2}\left[\tanh\frac{\Delta(g)+W(g)}{2T}\right]^N \tag{14}$$

where $\Delta(g)$ and $\Delta(g) + W(g)$ are respectively the minimum and maximum of the energy spectrum$\epsilon_{n,\vec{k}}[g]$ that depends on g. Since $\Delta(g)$ and $W(g)$ remains finite in the thermodynamic limit of $N \to \infty$ Eq.(14) implies

$$\lim_{N\to\infty}\langle\Phi_x\rangle \to 0 \tag{15}$$

in the thermodynamic limit. Further using $[\tanh(\Delta/2T)]^N \approx 1 - 2N\exp(-\Delta/T)$ at large N and low T, we find the system size dependence of the crossover temperature:

$$T^* \sim \frac{\Delta(g)}{\ln N}, \tag{16}$$